

A Short Note on Contracting Self-Similar Solutions of the Curve Shortening Flow

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Abstract

By the curve shortening flow, the only closed contracting self-similar solutions are circles: we give a very short and intuitive geometric proof of this basic and classical result using an idea of Gage [4].

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MSC 53C44, 53A04.

1 Introduction

Let γ be a smooth closed curve parametrized by arc length, embedded in the Euclidean plane endowed with its canonical inner product denoted by a single point. A one-parameter smooth family of plane closed curves $(\gamma(\cdot, t))_t$ with initial condition $\gamma(\cdot, 0) = \gamma$ is said to evolve by the curve shortening flow (CSF for short) if

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{n} \quad (1)$$

where κ is the signed curvature and \mathbf{n} the inward pointing unit normal. By the works of Gage, Hamilton and Grayson any embedded closed curve evolves to a convex curve (or remains convex if so) and shrinks to a point in finite time¹.

In this note, we are interested by *self-similar* solutions that is solutions which shapes change homothetically during the evolution. This condition is equivalent to say, after a suitable parametrization, that

$$\kappa = \varepsilon \gamma \cdot \mathbf{n} \quad (2)$$

with $\varepsilon = \pm 1$. If $\varepsilon = -1$ (resp. $+1$), the self-similar family is called *contracting* (resp. *expanding*). For instance, for any positive constant C the concentric circles $(s \mapsto \sqrt{1-2t}(\cos s, \sin s))_t$ form a self-similar contracting solution of the CSF shrinking to a point in finite time and as a matter of fact, there is no more example than this

¹The reader could find a dynamic illustration of this result on the internet page <http://a.carapetis.com/csf/>

one:

by the curve shortening flow, the only closed embedded contracting self-similar solutions are circles².

To the author knowledge, the shortest proof of this was given by Chou-Zhu [3] by evaluating a clever integral. The proof given here is purely geometric and based on an genuine trick used by Gage in [4].

2 A geometric proof

Let γ be a closed, simple embedded plane curve, parametrized by arclength s , with signed curvature κ . By reversing the orientation if necessary, we can assume that the curve is counter-clockwise oriented. The length of γ is denoted by L , the compact domain enclosed by γ will be denoted by Ω with area A and the associated moving Frenet frame by (\mathbf{t}, \mathbf{n}) . Let $\gamma_t = \gamma(t, \cdot)$ be the one parameter smooth family solution of the CSF, with the initial condition $\gamma_0 = \gamma$.

Multiplying (1) by \mathbf{n} , we obtain

$$\frac{\partial \gamma}{\partial t} \cdot \mathbf{n} = \kappa \quad (3)$$

Equations (1) and (3) are equivalent: from (3), one can look at a reparametrization $(t, s) \mapsto \varphi(t, s)$ such that $\tilde{\gamma}(t, s) = \gamma(t, \varphi(t, s))$ satisfies (1). A simple calculation leads to an ode on φ which existence is therefore guaranteed [3]. From now, we will deal with equation (3).

If a solution γ of (3) is self-similar, then there exists a non-vanishing smooth function $t \mapsto \lambda(t)$ such that $\gamma_t(s) = \lambda(t) \gamma(s)$. By (2), this leads to $\lambda'(t) \gamma(s) \cdot \mathbf{n}(\gamma_t(s)) = \kappa(\gamma_t(s))$, that is $\lambda'(t) \lambda(t) \gamma(s) \cdot \mathbf{n}(s) = \kappa(s)$. The function $s \mapsto \gamma(s) \cdot \mathbf{n}(s)$ must be non zero at some point (otherwise κ would vanish everywhere and γ would be a line) so the function $\lambda' \lambda$ is constant equal to a real ε which can not be zero. By considering the new curve $s \mapsto \sqrt{|\varepsilon|} \gamma(s / \sqrt{|\varepsilon|})$ which is still parametrized by arc length, we can assume that $\varepsilon = \pm 1$. In the sequel we will assume that γ is contracting, that is $\varepsilon = -1$ which says that we have the fundamental relation:

$$\kappa + \gamma \cdot \mathbf{n} = 0 \quad (4)$$

An immediate consequence is the value of A : indeed, by the divergence theorem and the turning tangent theorem,

$$A = \frac{1}{2} \int_{\gamma} (x dy - y dx) = -\frac{1}{2} \int_0^L \gamma(s) \cdot \mathbf{n}(s) ds = \frac{1}{2} \int_0^L k(s) ds = \pi$$

²The nonembedded closed curves were studied and classified by Abresch and Langer[1]

Therefore, our aim will be to prove that $L = 2\pi$ and we will conclude by using the equality case in the isoperimetric inequality.

The second remark is that the curve is an oval or strictly convex: indeed, by differentiating (4) and using Frenet formulae, we obtain that $\kappa' = \kappa \gamma \cdot \gamma'$ which implies that $\kappa = Ce^{|\gamma|^2/2}$ for some non zero constant C . As the rotation index is $+1$, C is positive and so is κ .

2.1 Polar tangential coordinates

As equation (4) is invariant under Euclidean motions, we can assume that the origin O of the Euclidean frame lies within Ω with axis $[Ox]$ meeting γ orthogonally. We introduce the angle function θ formed by $-\mathbf{n}$ with the x -axis as shown in the figure below:

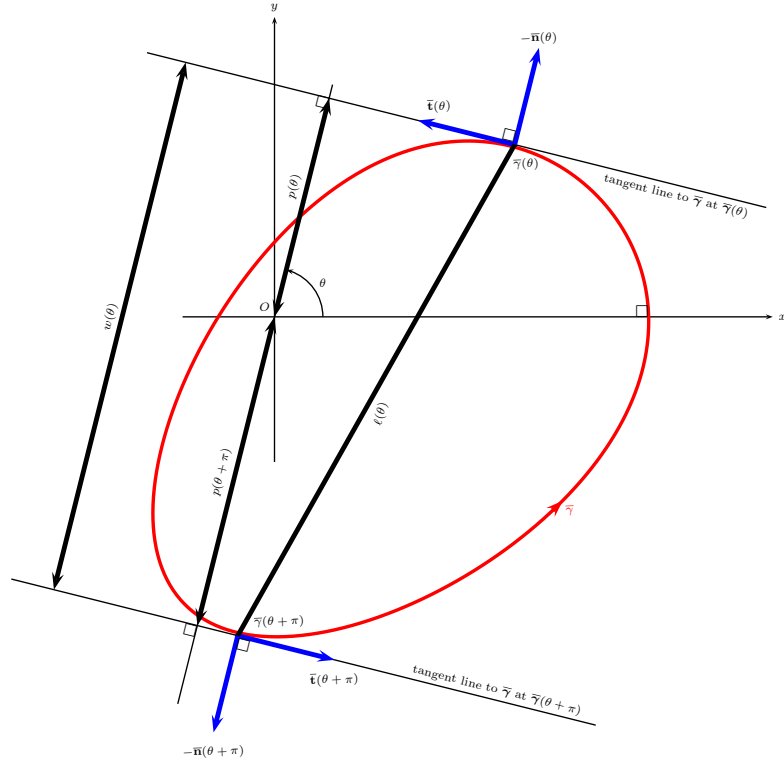


Figure 1: Polar tangential coordinates

Then $-\mathbf{n}(s) = (\cos \theta(s), \sin \theta(s))$. Differentiating this equality, we obtain $\kappa(s) \mathbf{t}(s) = \theta'(s) \mathbf{t}(s)$, that is $\theta(s) = \int_0^s \kappa(u) du$ since $\theta(0) = 0$. As $\theta' = \kappa \geq C > 0$, θ is a strictly increasing function on \mathbb{R} onto \mathbb{R} . So θ can be choosen as a new parameter and we set $\bar{\gamma}(\theta) = (\bar{x}(\theta), \bar{y}(\theta)) = \gamma(s)$, $\bar{\mathbf{t}}(\theta) = (-\sin \theta, \cos \theta)$, $\bar{\mathbf{n}}(\theta) = (-\cos \theta, -\sin \theta)$ and we consider the function p defined by $p(\theta) = -\bar{\gamma}(\theta) \cdot \bar{\mathbf{n}}(\theta)$. As $\theta(s+L) = \theta(s) + 2\pi$, we note that $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$ and p are 2π -periodic functions. The curve $\bar{\gamma}$ is regular but not necessarily parametrized by arc length because $\bar{\gamma}'(\theta) = \frac{1}{k(s)} \gamma'(s)$ and we note $\bar{\kappa}$ its curvature. By definition, we have

$$\bar{x}(\theta) \cos \theta + \bar{y}(\theta) \sin \theta = p(\theta) \quad (5)$$

which, by differentiation w.r.t. θ , gives

$$-\bar{x}(\theta) \sin \theta + \bar{y}(\theta) \cos \theta = p'(\theta) \quad (6)$$

Thus,

$$\begin{cases} \bar{x}(\theta) &= p(\theta) \cos \theta - p'(\theta) \sin \theta \\ \bar{y}(\theta) &= p(\theta) \sin \theta + p'(\theta) \cos \theta \end{cases} \quad (7)$$

Differentiating once more, we obtain

$$\begin{cases} \bar{x}'(\theta) &= -[p(\theta) + p''(\theta)] \sin \theta \\ \bar{y}'(\theta) &= [p(\theta) + p''(\theta)] \cos \theta \end{cases} \quad (8)$$

Since γ is counter-clockwise oriented, we have $p + p'' > 0$.

Coordinates $(\theta, p(\theta))_{0 \leq \theta \leq 2\pi}$ are called *polar tangential coordinates* and p is the *Minkowski support function*. By (8), we remark that the tangent vectors at $\bar{\gamma}(\theta)$ and $\bar{\gamma}(\theta + \pi)$ are parallel. We will introduce the *width function* w defined by

$$w(\theta) = p(\theta) + p(\theta + \pi)$$

which is the distance between the parallel tangent lines at $\bar{\gamma}(\theta)$ and $\bar{\gamma}(\theta + \pi)$ and we denote by $\ell(\theta)$ the segment joining $\bar{\gamma}(\theta)$ and $\bar{\gamma}(\theta + \pi)$.

With these coordinates, the perimeter has a nice expression:

$$L = \int_0^{2\pi} (\bar{x}^2 + \bar{y}^2)^{1/2} d\theta = \int_0^{2\pi} (p + p'') d\theta = \int_0^{2\pi} p d\theta \quad (\text{Cauchy formula})$$

The curvature $\bar{\kappa}$ of $\bar{\gamma}$ is

$$\bar{\kappa} = \frac{\bar{x}'\bar{y}'' - \bar{x}''\bar{y}'}{(\bar{x}^2 + \bar{y}^2)^{3/2}} = \frac{1}{p + p''}$$

and equation (4) reads $\bar{\kappa} = p$. So, finally,

$$\bar{\kappa} = p = \frac{1}{p + p''} \quad (9)$$

2.2 Bonnesen inequality

If B is the unit ball of the Euclidean plane, it is a classical fact that the area of $\Omega - tB$ (figure 2) is $A_\Omega(t) = A - Lt + \pi t^2$ [2].

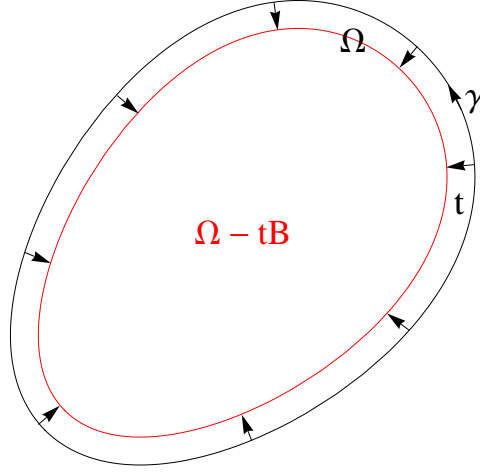


Figure 2: The domain $\Omega - tB$ with positive t

The roots t_1, t_2 (with $t_1 \leq t_2$) of $A_\Omega(t)$ are real by the isoperimetric inequality and they have a geometric meaning: indeed, if R is the circumradius of Ω , that is the radius of the circumscribed circle, and if r is the inradius of Ω , that is the radius of the inscribed circle, Bonnesen [2, 5] proved in the 1920's a series of inequalities, one of them being the following one:

$$t_1 \leq r \leq R \leq t_2$$

Moreover, and this is a key point in the proof, any equality holds if and only if $\bar{\gamma}$ is a circle. We also note that $A_\Omega(t) < 0$ for any $t \in (t_1, t_2)$.

2.3 End of proof

Special case: $\bar{\gamma}$ is symmetric w.r.t. the origin O , that is $\bar{\gamma}(\theta + \pi) = -\bar{\gamma}(\theta)$ for all $\theta \in [0, 2\pi]$, which also means that $p(\theta + \pi) = p(\theta)$ for all $\theta \in [0, 2\pi]$. So the width function w is twice the support function p . As $2r \leq w \leq 2R$, we deduce that for all θ , $r \leq p(\theta) \leq R$. If $\bar{\gamma}$ is not a circle, then one would derive from Bonnesen inequality that $t_1 < r \leq p(\theta) \leq R < t_2$. So $A_\Omega(p(\theta)) < 0$ for all θ , that is $\pi p^2(\theta) < Lp(\theta) - \pi$. Multiplying this inequality by $1/p = p + p'' (> 0)$ and integrating on $[0, 2\pi]$, we would obtain $\pi L < \pi L$ by Cauchy formula ! By this way, we proved that any symmetric smooth closed curve satisfying (4) is a circle. As the area is π , the length is 2π of course.

General case: using a genuine trick introduced by Gage [4], we assert that

for any oval enclosing a domain of area A , there is a segment $\ell(\theta_0)$ dividing Ω into two subdomains of equal area $A/2$.

Proof: let $\sigma(\theta)$ be the area of the subdomain of Ω , bounded by $\bar{\gamma}([\theta, \theta + \pi])$ and the segment $\ell(\theta)$. We observe that $\sigma(\theta) + \sigma(\theta + \pi) = A$. We can assume without loss of generality that $\sigma(0) \leq A/2$. Then we must have $\sigma(\pi) \geq A/2$, and by continuity of σ and the intermediate value theorem, there exists θ_0 such that $\sigma(\theta_0) = A/2$ and the segment $\ell(\theta_0)$ proves the claim. \square

Let ω_0 be the center of $\ell(\theta_0)$. If $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the two arcs of $\bar{\gamma}$ separated by $\ell(\theta_0)$, we denote by $\bar{\gamma}_i^s$ ($i = 1, 2$) the closed curve formed by $\bar{\gamma}_i$ and its reflection through ω_0 . Each $\bar{\gamma}_i^s$ is a symmetric closed curve and as $\ell(\theta_0)$ joins points of the curve where the tangent vectors are parallel, each one is strictly convex and smooth (figure 3).

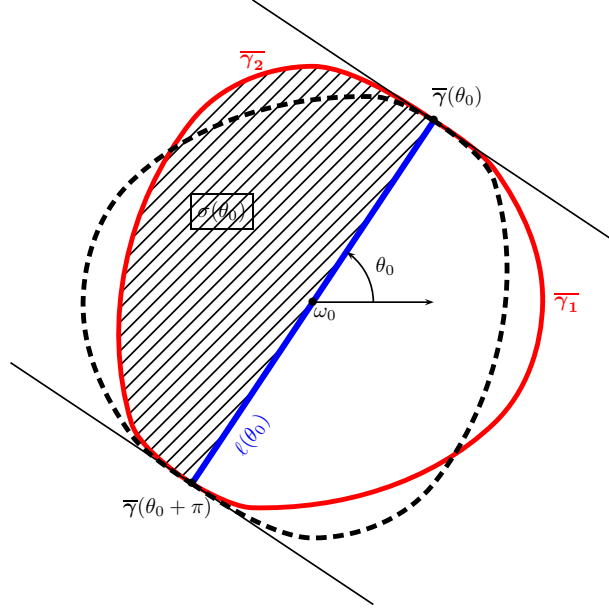


Figure 3: Symmetrization of the curve

Moreover, each $\bar{\gamma}_i^s$ satisfies equation (4) and encloses a domain of area $2 \times A/2 = A$. So we can apply the previous case to these both curves and this gives that $\text{length}(\bar{\gamma}_i^s) = 2\pi$ for $i = 1, 2$, that is $\text{length}(\bar{\gamma}_i) = \pi$ which in turn implies that $L = 2\pi$. So $\bar{\gamma}$ (that is γ) is a circle. This proves the theorem. \square

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